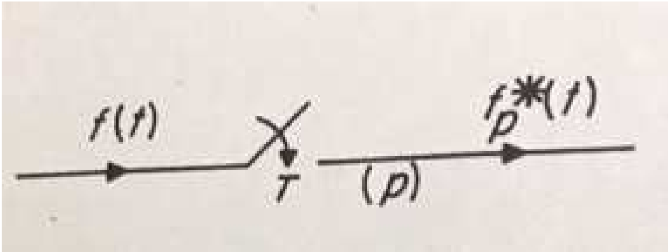
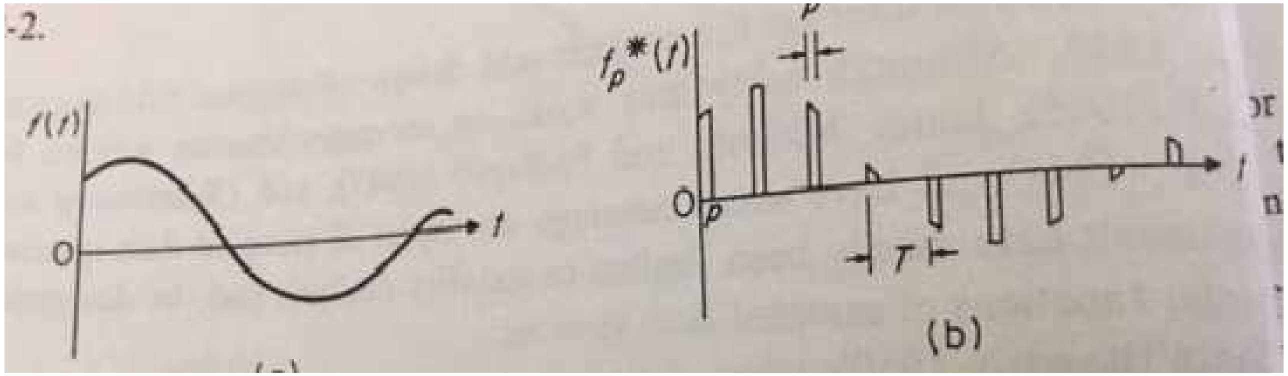
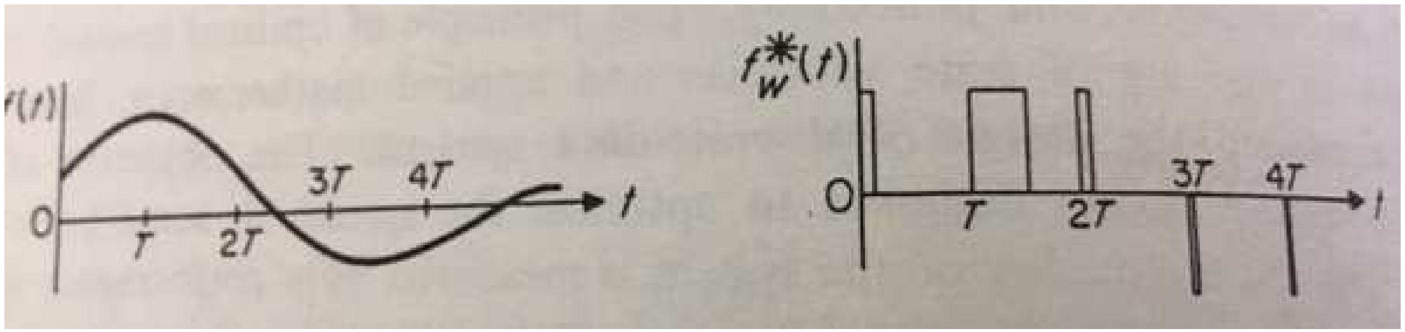
Introduction to Digital Control

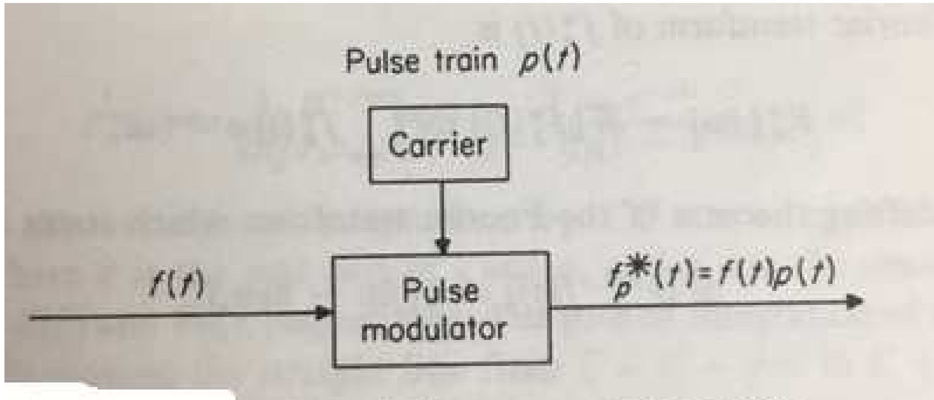
• Sampling and Data reconstruction Processes



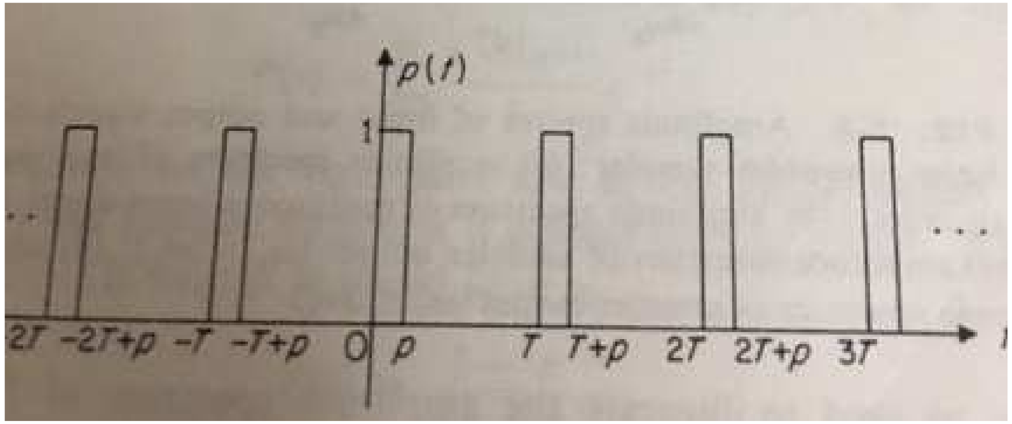




Mathematical Description of The Uniform Rate Sampling Process



Pulse modulator as a sampling device.



$$p(t) = \sum_{k=-\infty}^{\infty} [u(t - kT) - u(t - kT - p)] \quad (p < T)$$

$$f_p^*(t) = f(t) \times p(t)$$

$$f_p^*(t) = f(t) \sum_{k=-\infty}^{\infty} [u(t - kT) - u(t - kT - p)] \quad (p < T)$$

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \qquad \omega_s = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_0^T p(t)\, e^{-jn\omega_s t}\, dt$$

$p(t) = 1 \quad 0 \le t \le p$ otherwise $= 0$

$$c_n = \frac{1}{T}\int_0^p p(t)\, e^{-jn\omega_s t}\, dt$$

$$= \frac{1 - e^{-jn\omega_s p}}{jn\omega_s T}$$

$$= \frac{e^{+jn\omega_s(p/2 - p/2)} - e^{-jn\omega_s(p/2 + p/2)}}{jn\omega_s T}$$

$$= 2\frac{(\sin jn\omega_s p/2)}{n\omega_s T} e^{-jn\omega_s p/2}$$

$$= p/T \; \frac{\sin(n\omega_s p/2)}{n\omega_s p/2} e^{-jn\omega_s p/2}$$

$$p(t) = p/T \sum_{n=-\infty}^{\infty} \frac{\sin(n\omega_s p/2)}{n\omega_s p/2} e^{-jn\omega_s p/2}$$

Fourier transform of $f_p^*(t)$

$$F_p^*(j\omega) = \int_{-\infty}^{\infty} f_p^*(t)\, e^{-j\omega t}\, dt$$

using shifting theorem

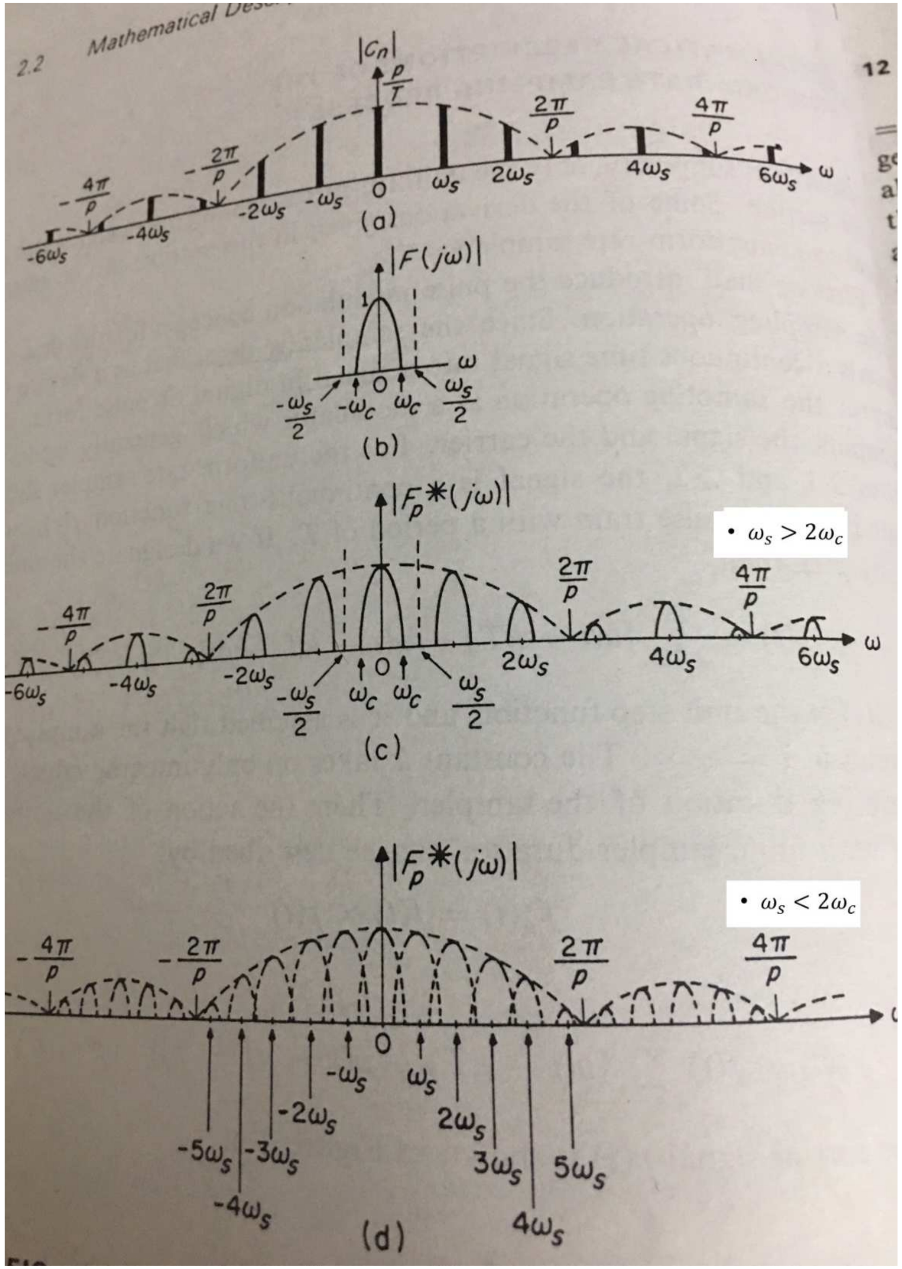
$$\mathcal{F}\, e^{jn\omega_s t} f(t) = F(j\omega - jn\omega_s)$$

$$F_p^*(j\omega) = \sum_{n=-\infty}^{\infty} c_n F(j\omega - jn\omega_s)$$

$n \to 0$

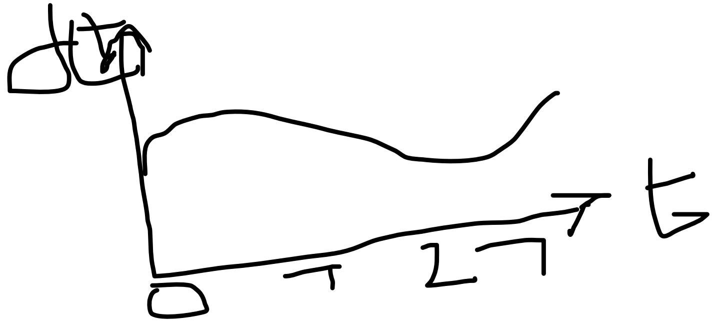
$$F_p^*(j\omega) = c_0 F(j\omega) = p/T \, F(j\omega)$$

$$c_0 = \lim_{n \to 0} c_n = p/T$$



2.2 Mathematical Des
12
$|c_n|$
$\frac{p}{T}$
$\frac{2\pi}{p}$
$\frac{4\pi}{p}$
$-\frac{2\pi}{p}$
$-\frac{4\pi}{p}$
$-6\omega_s$
$-4\omega_s$
$-2\omega_s$
$-\omega_s$
0
ω_s
$2\omega_s$
$4\omega_s$
$6\omega_s$
ω
(a)
$|F(j\omega)|$
1
$-\frac{\omega_s}{2}$
$-\omega_c$
ω_c
$\frac{\omega_s}{2}$
(b)
$|F_p^*(j\omega)|$
• $\omega_s > 2\omega_c$
(c)
• $\omega_s < 2\omega_c$
$-5\omega_s$
$-3\omega_s$
$3\omega_s$
$5\omega_s$
(d)

Flat Top approximation and the Ideal Sampler



$f_p^*(t) = f(kT) for\ kT \leq t < kT + p$

= 0 kT+p≤t<(k+1)T

- Where k=0,1,2,3…..
- $f_p^*(t) = \sum_{k=0}^{\infty} f(kT)(u(t-kT) - u(t-KT-p)$
- Taking Laplace transform

$$F_p^*(s) = \sum_{k=0}^{\infty} f(kT)\left(\frac{1-e^{-ps}}{s}\right)e^{-kTs}$$

$$\frac{1-e^{-ps}}{s} = \frac{1}{s}\left(1-\left(1-ps+\frac{(ps)^2}{2!}-\cdots\right)\right)$$

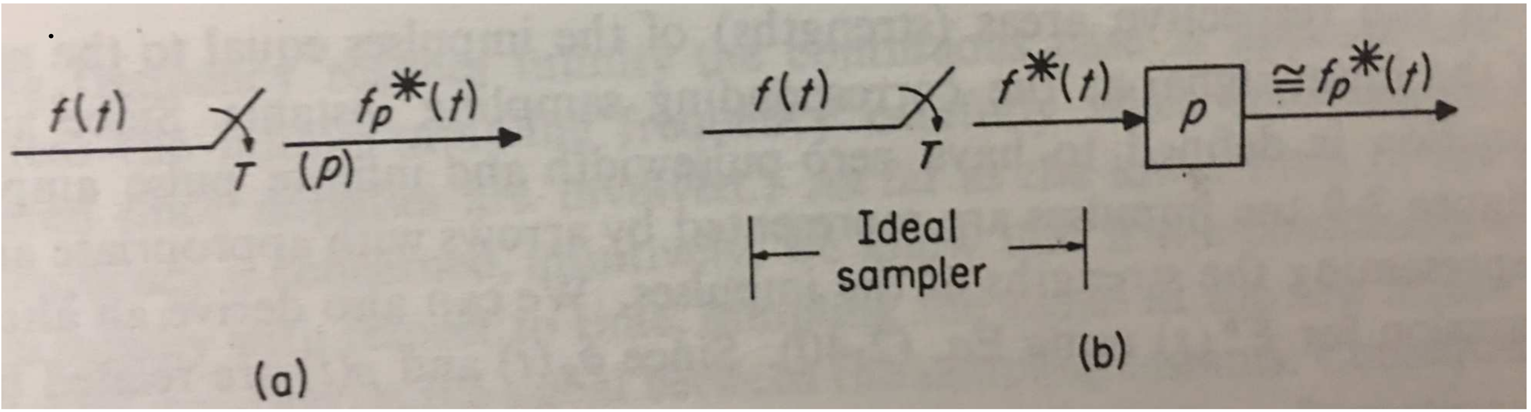
$$\simeq \frac{1}{s}\left(1 - (1 - ps)\right)$$

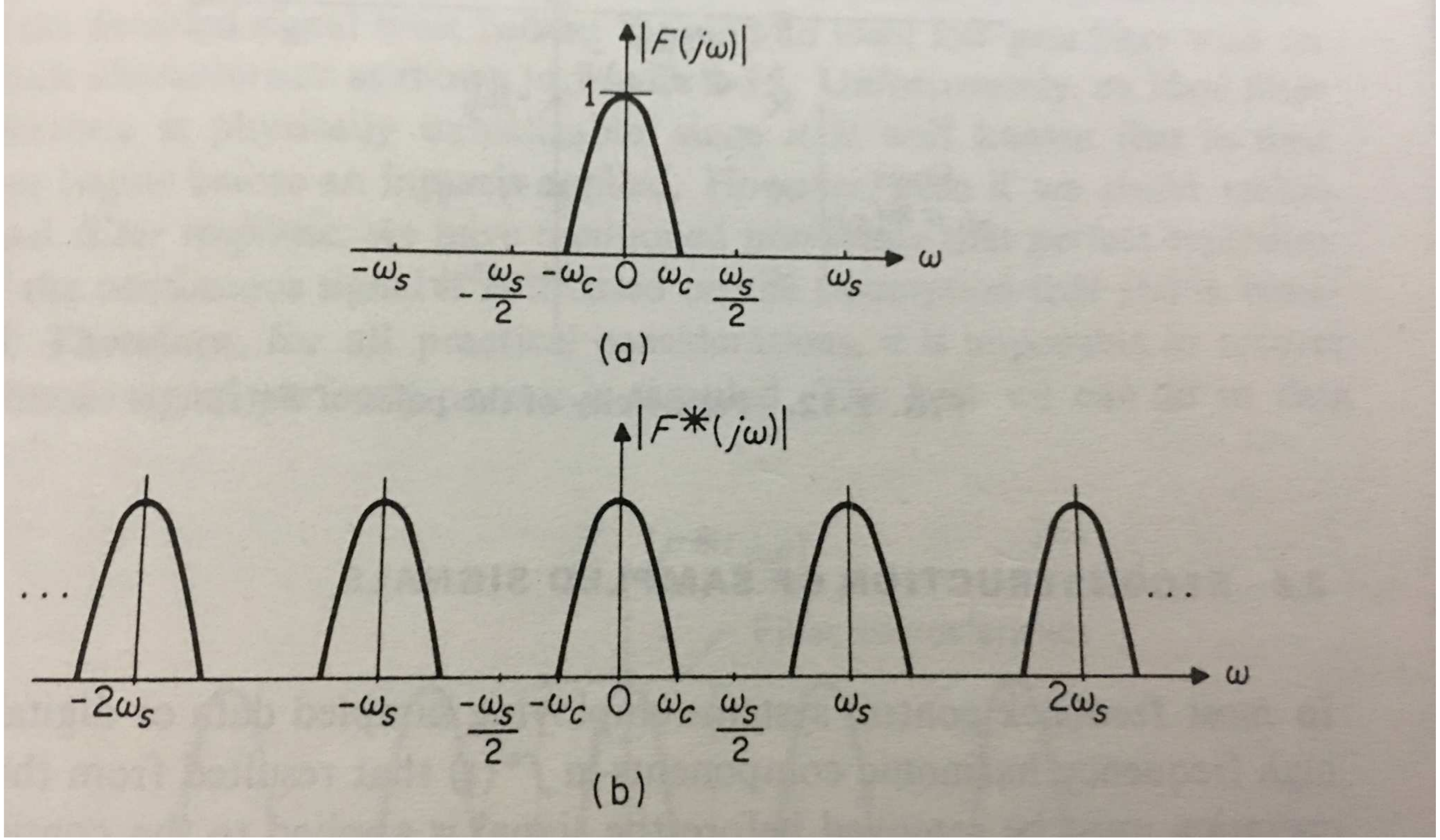
$$\simeq \frac{1}{s}\left(1 - 1 + ps\right)$$

$$\simeq p$$

$$F_p^*(s) \approx p \sum_{k=0}^{\infty} f(kT) e^{-kTs}$$

$f_p^*(t) \approx p \sum_{k=0}^{\infty} f(kT)\delta(t - kT)$

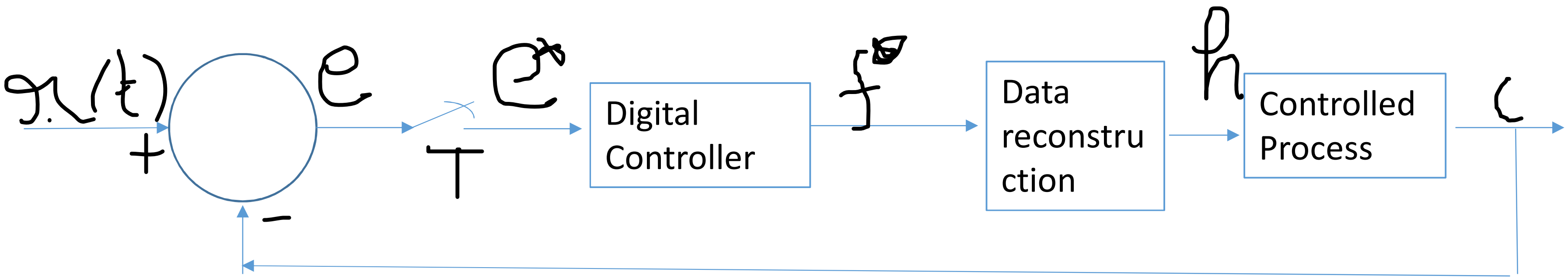


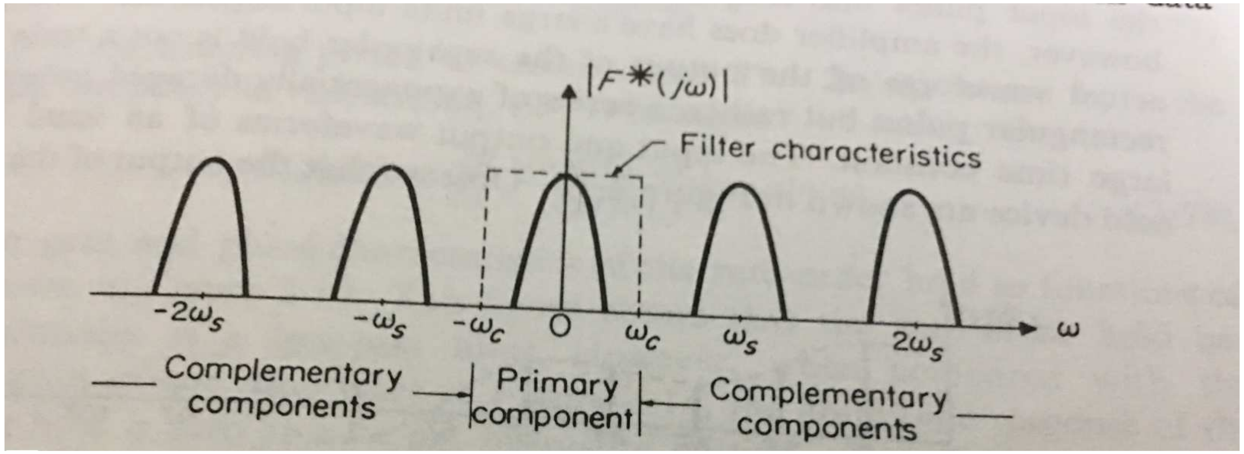


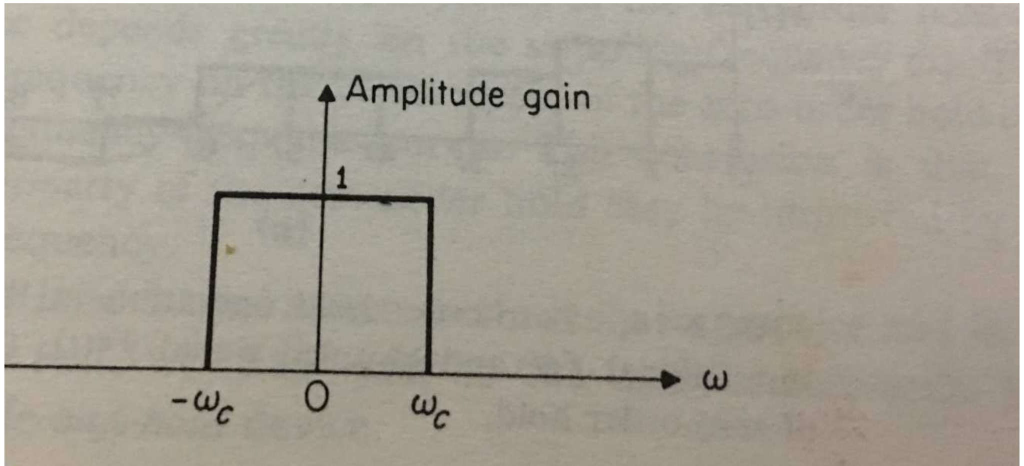
|F(jω)|
1
-ωs
-ωs/2
-ωc
0
ωc
ωs/2
ωs
ω
(a)
|F*(jω)|
. . .
-2ωs
-ωs
-ωs/2
-ωc
0
ωc
ωs/2
ωs
2ωs
ω
(b)

Sampling Theorem states that if a signal contains no frequency higher than ω_s rad/sec it is completely characterized by the value of signals measured at instants of time separated by $T=1/2(2\pi/\omega_c)$ sec.

Reconstruction of Sampled Signals (data reconstruction device(DRD))







Amplitude characteristic of an ideal filter.

- Ideal filter characteristics are physically unrealizable
- Because its time response begins before i/p is applied
- Even if we realize it perfect reproduction of continuous time signal is based on the assumption hat f(t) is band limited
- Therefore perfect recovery of continuous time signal is not possible practically
- Try to approximate as close as possible
- DRD involves compromise between requirement of stability and accuracy

- Sequence of numbers f(0),f(T),f(2T),……f(kT),..or train of impulses with the strength of the impulse occurring at t=kT equal to f(kT) k=0,1,2,…..
- Continuous signal must f(t), t≥0 be reconstructed from the information available
- Data reconstruction process may be regarded as an extrapolation process
- Because signal is to be constructed based on information available at past sampling instants
- e.g., f(t) between f(kT) and f(k+1)T to be estimated based on values at preceeding intervals kT,(k-1)T,f(2k-T),….f(0)

- A well known method of generating this desired approximation is base on power series expansion of f(t) in the interval between sampling instants kT and (k+1)T i.e.,

- $f_k(t)=f(kT)+ f^{(1)}(kT)(t-kT)+f^{(2)}(kT)(t-kT)^2/2!$...........

- Where $f_k(t)=f(t)$ for $kT \le t<(k+1)T$

- $f^{(1)}(kT) = df(t)/dt|_{t=kT}$
- In order to evaluate the coefficients of the series derivatives must be obtained at the sampling instants
- Simple expression involving only two data points gives an estimate of first derivative

$$f^{(1)}(kT) = \frac{1}{T}\{f(kT) - f(k-1)T\}$$

$$f^{(2)}(kT) = \frac{1}{T}\{f^{(1)}(kT) - f^{(1)}(k-1)T\}$$

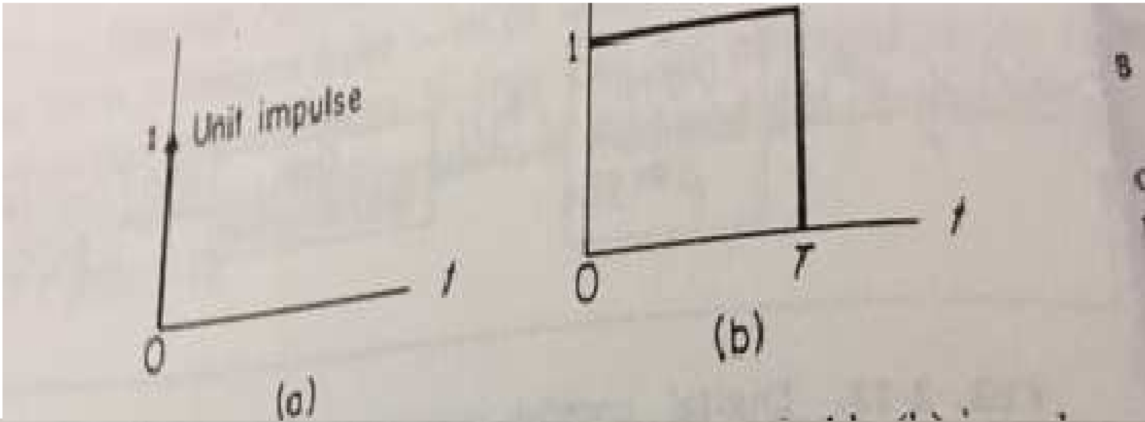
$$= \frac{1}{T}\left\{\frac{1}{T}\{f(kT) - f(k-1)T\} - \frac{1}{T}[f(k-1)T - f(k-2)T]\right.$$

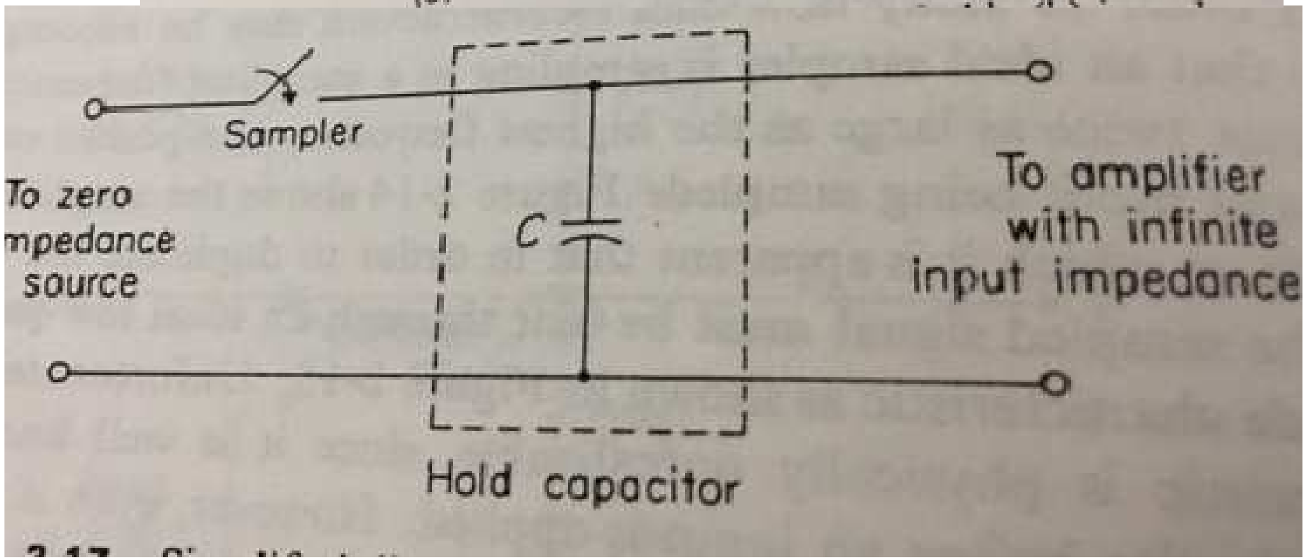
$$= \frac{f(kT) - 2f(k-1)T + f(k-2)T}{T^2}$$

$$f^{(3)}(kT) = \frac{1}{T}\{f^{(2)}(kT) - f^{(2)}(k-1)T\}$$

- **<u>Zero order hold</u>**
- $f_k(t)=f(kT)$

a) Unit impulse Input
b) Impulse response of zero order hold





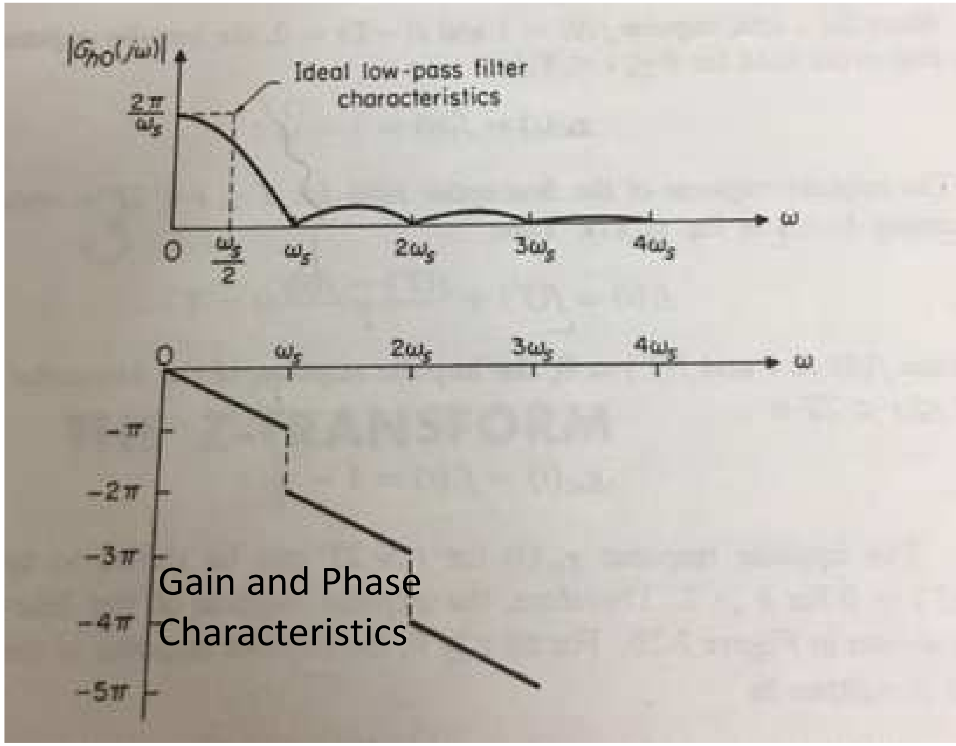
$g_{ho}(t) = u(t) - u(t-T)$

Tr.fn of Z.O.H

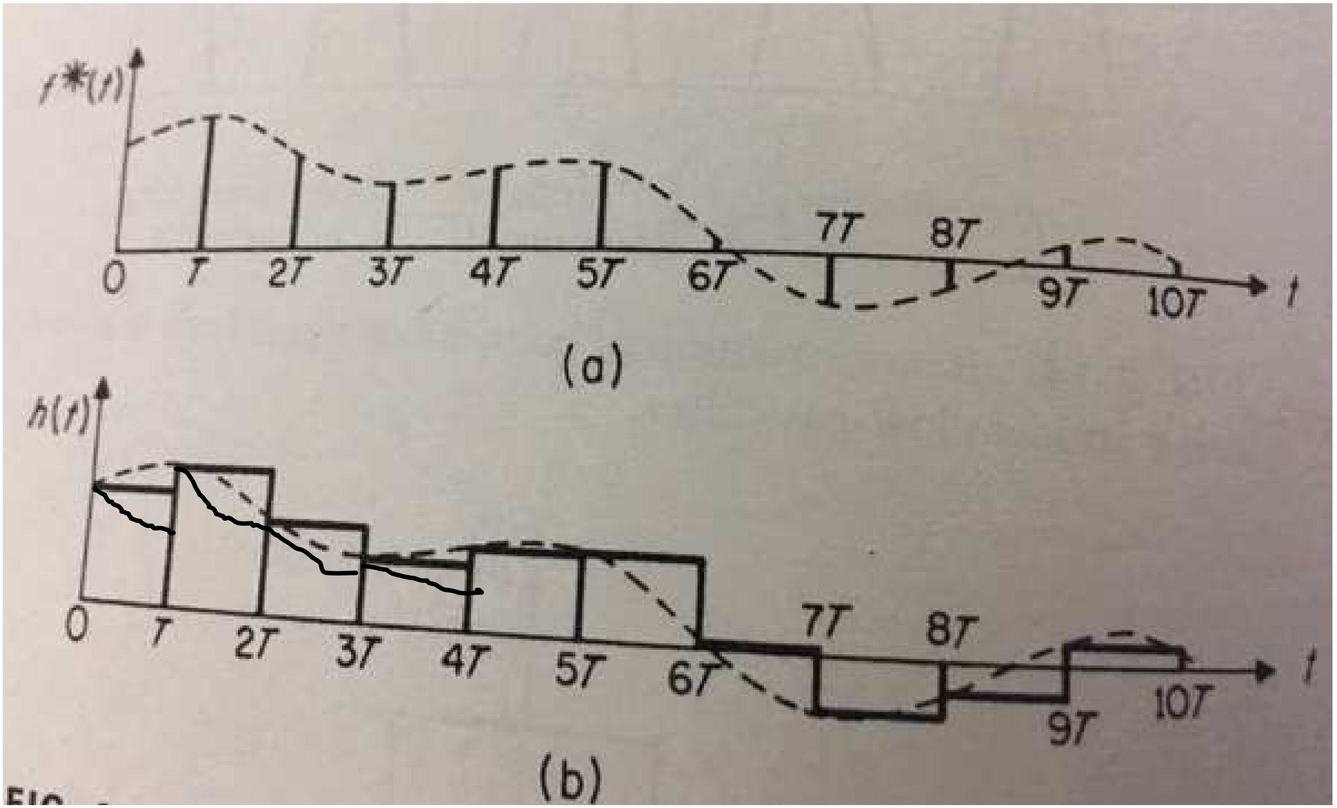
$$G_{ho}(S) = \frac{1-e^{-TS}}{S}$$

$$G_{ho}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = \frac{2\sin(\omega T/2)\, e^{-j\omega T/2}}{\omega}$$

$$G_{ho}(j\omega) = T\,\frac{\sin(\omega T/2)}{(\omega T/2)}\, e^{-j\omega T/2}$$



Gain and Phase Characteristics


f*(t)
0
T
2T
3T
4T
5T
6T
7T
8T
9T
10T
t
(a)
h(t)
0
T
2T
3T
4T
5T
6T
7T
8T
9T
10T
t
(b)

First order hold

$f_K(t) = f(KT) + f^{(1)}(KT)(t-KT)$

$KT \le t \le (K+1)T$

$f_K(t) = f(KT) + \frac{f(KT) - f((K-1)T)}{T}(t-KT)$

impulse response:

$f_0(t) = f(0) + \frac{f(0) - f(-T)}{T} t$

$0 \le t < T$

$f(0) = 1$ + $f(-T) = 0$

$g_{h1}(t) = f_0(t) = 1 + \frac{1}{T} t$

$T \le t < 2T$ $K = 1$

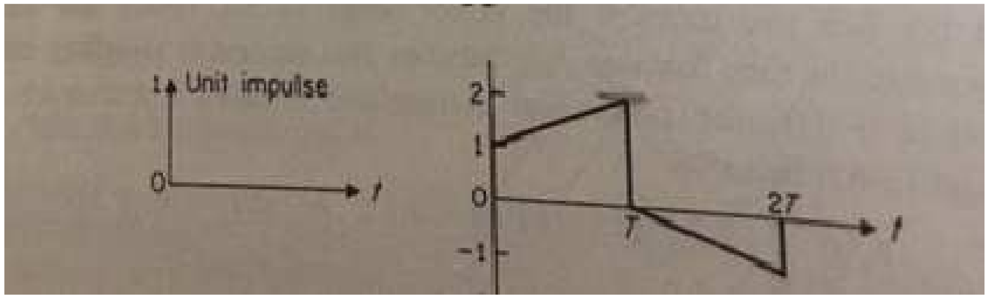
$f_1(t) = f(T) + \frac{f(T) - f(0)}{T}(t-T)$

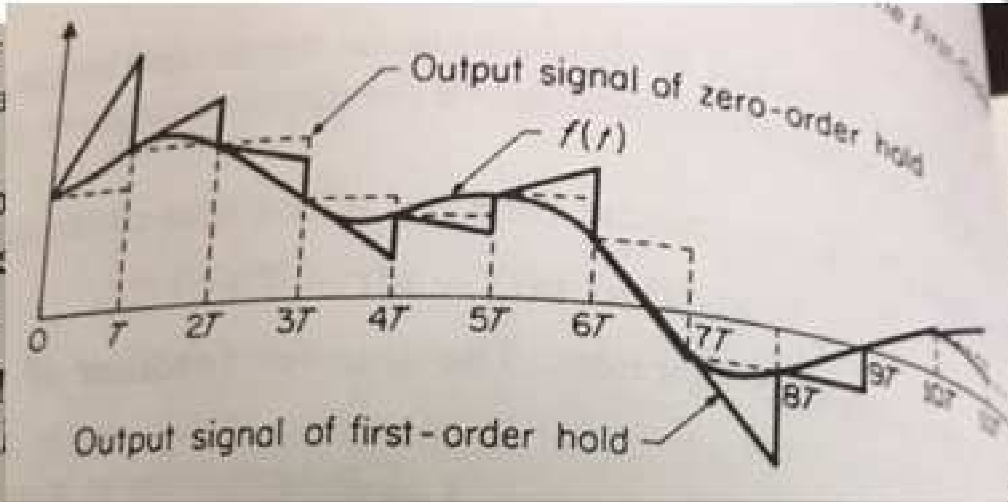
$f(0) = 1$ $f(T) = 0$

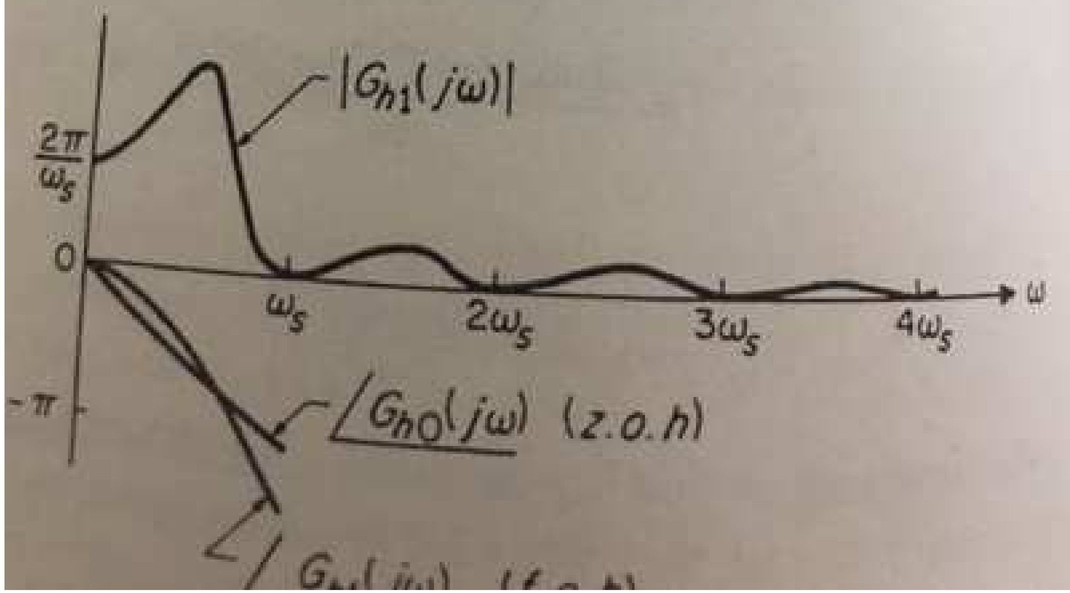
$g_{h1}(t) = f_1(t) = 1 - \frac{1}{T} t$

• $g_{h1}(t) = (1 + \frac{t}{T})u(t) - 2(1 + \frac{t-T}{T})u(t-T$

$+ \{1 + \frac{t-2T}{T})u(t-2T)$




Output signal of zero-order hold
f(t)
0
T
2T
3T
4T
5T
6T
7T
8T
9T
10T
Output signal of first-order hold


|G_h1(jω)|
2π/ω_s
0
ω_s
2ω_s
3ω_s
4ω_s
ω
−π
∠G_h0(jω) (z.o.h)

Anti-Aliasing Filter

- Signals in control system have frequency spectra consisting of low frequency components as well as high frequency noise components
- All signals higher than $\omega_s/2$ appear as signals of frequencies between 0 and $\omega_s/2$ due aliasing effect
- To avoid aliasing proper choice of sampling frequency has to be made or use analog filter before sampling.

Practical Aspects of the choice of Sampling Theorem

Every time a digital system is designed choice of sampling theorem has to be made

- Long sampling interval

reduce computational load and h/w cost

But degrading effects start to become significant

- Lower the limit of sampling interval

Increases accuracy but increases cost of h/w

Therefore tradeoff to be made between accuracy and h/w cost

Limiting Factors for the choice of Sampling Rate

- Information loss due to sampling

- Information loss due to disturbances
- Destabilizing Effects
- Algorithm – Accuracy effects
- Word-length effects

- Empirical Rule for the Selection of Sampling Rate

General SR for process industry flow-1-3 sec,level 5-10sec, pressure 1-5 sec, temperature 10-20 sec

Fast acting electromechanical system require short SI few msec

Thumb rule SF shorter than time constant , =1/10 of smallest time constant or 1/ largest real pole

Reasonable sampling rates are 10 to 30 times the BW

$T \leq 1/10\ T_s$