The Z-Transform

- Z-transform similar to Laplace transforms
- One of the mathematical tool for analysis and design of Linear single variable Digital system with uniform sampling
- When the system is multi variable, non linear, time varying, non uniform sampling difficulties are encountered then we move to state variable approach,

- $F^*(s) \approx \sum_{k=0}^{\infty} f(kT)e^{-kTs}$

- e^{-kTs} not a rational function of s
- Transform irrational function into a rational function

F(z) where z and s are related

z=e^{Ts}

$$s = \frac{1}{T}\ln z$$

T is a sampling period and z is a complex variable

- $F^*(s = \frac{1}{T}\ln z)$=F(z)=$\sum_{k=0}^{\infty} f(kT) z^{-k}$

F(z)=

$$\sum f(t)$$

$$F(z) = \mathcal{L}[f^*(t)]\Big|_{s=\frac{1}{T}\ln z} = [F^*(s)]_{s=\frac{1}{T}\ln z}$$

In summary z-transform involves following three steps

1. f(t) is sampled to give $f^*(t)$
2. The laplace transform of $f^*(t)$ is taken and

$$F^*(s) \approx \sum_{k=0}^{\infty} f(kT) e^{-kTs}$$

3. Replace e^{Ts} by z and

F(z)=$\sum_{k=0}^{\infty} f(kT) z^{-k}$

$$F^*(s) = \sum_{k=0}^{\infty} f(kT)e^{-kTs}$$

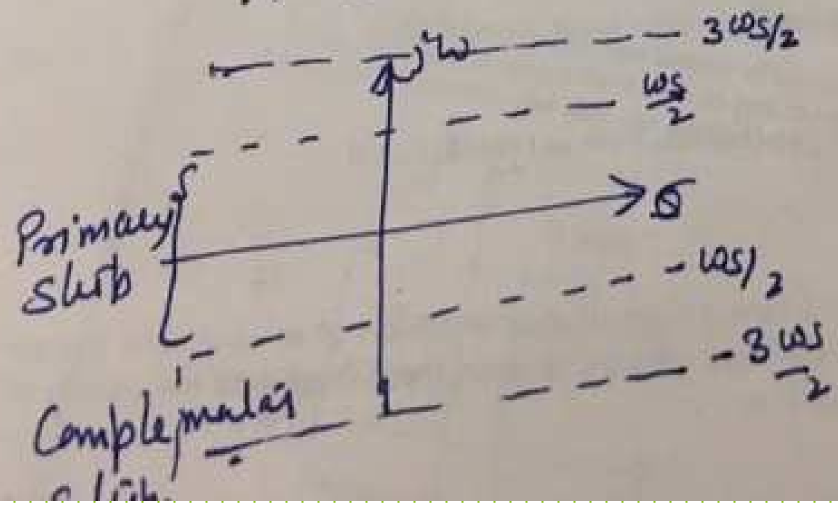
$$s = s + jn\omega_s$$

$$F^*(s+jn\omega_s) = \sum_{k=0}^{\infty} f(kT)\, e^{-kT(s+jn\omega_s)} = \sum_{k=0}^{\infty} f(kT)e^{-kTs} = F^*(s)$$

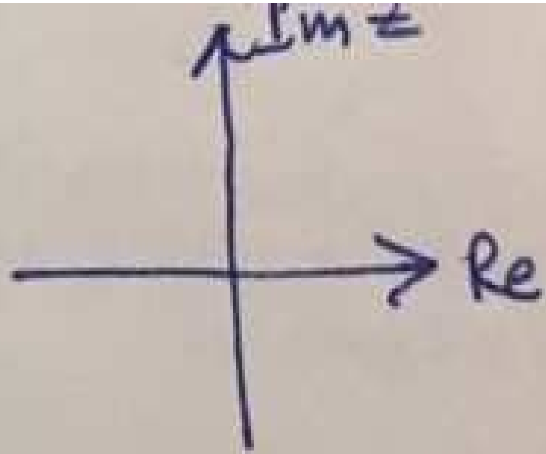
$$e^{-kT(s+jn\omega_s)} = e^{-kTs}\, e^{-kTjn\omega_s} = e^{-kTs}$$

$$e^{-kTjn\omega_s} = e^{-jnkT\frac{2\pi}{T}} = 1$$

$$\therefore\ e^{kT(s+jn\omega_s)} = z$$



$$
\begin{aligned}
z &= e^{sT} \\
&= e^{(\sigma \pm j\omega)T} \\
&= e^{\sigma T}\left[e^{\pm j\omega T}\right] \\
&= e^{\sigma T}\left\{\cos\omega T \pm j\sin\omega T\right) \\
&= r\cos\omega T \pm j r\sin\omega T \\
&= \mathrm{Re}\, z \pm j\, \mathrm{Im}(z)
\end{aligned}
$$



$r = e$

$s = \sigma \pm j\omega$

$\sigma = 0$

$s = \pm j\omega$

$z = e^{\pm j\omega T}$

$\sigma > 0$

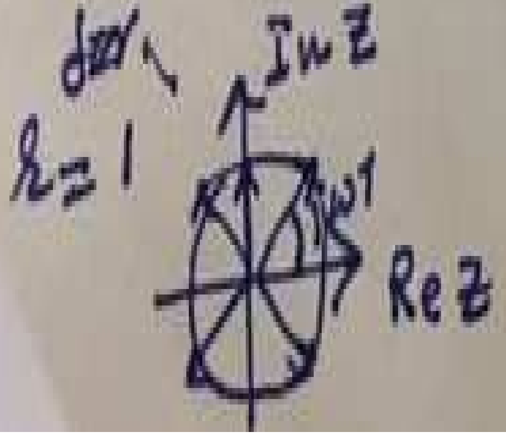
$s = \sigma \pm j\omega T$

$z = e^{\sigma T} e^{\pm j\omega\omega T}$

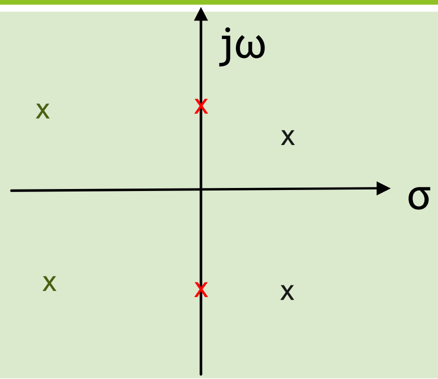
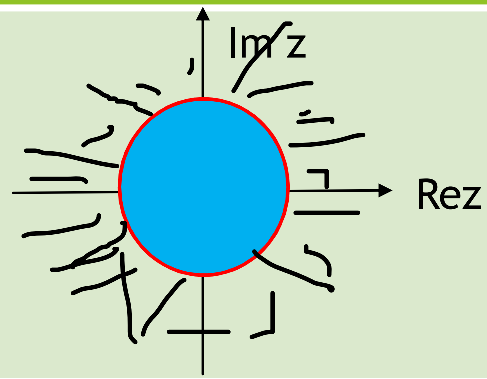
$\sigma < 0$

$s = -\sigma \pm j\omega T$

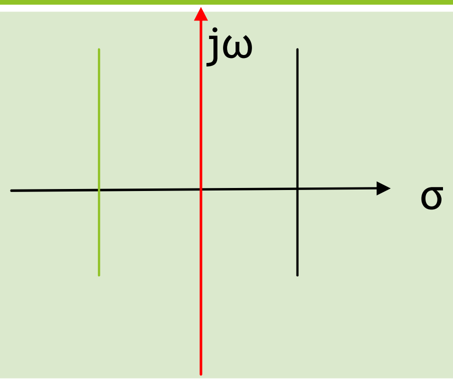
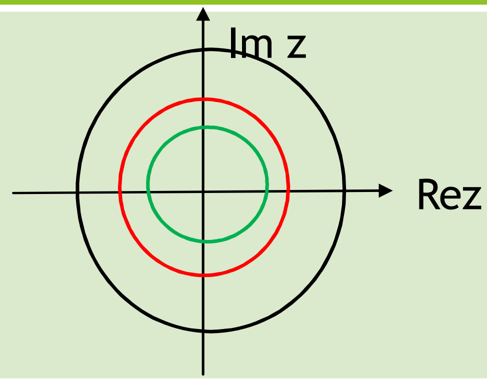
$z = e^{-\sigma T} e^{\pm j\omega T}$



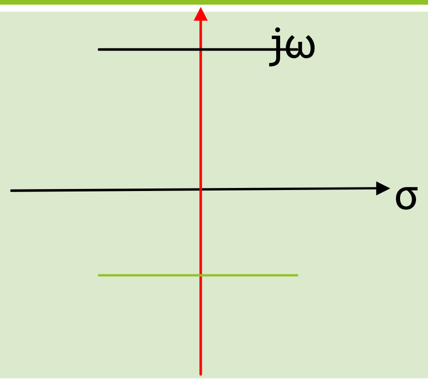
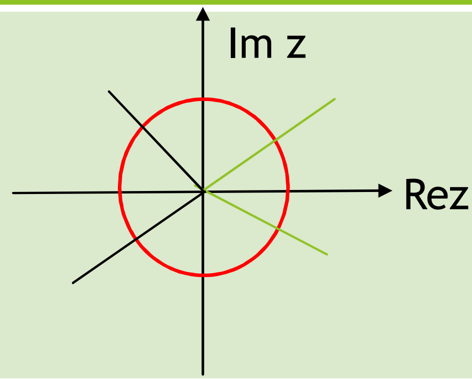
Mapping between s-plane and z-plane

S-Plane	Z-Plane
	

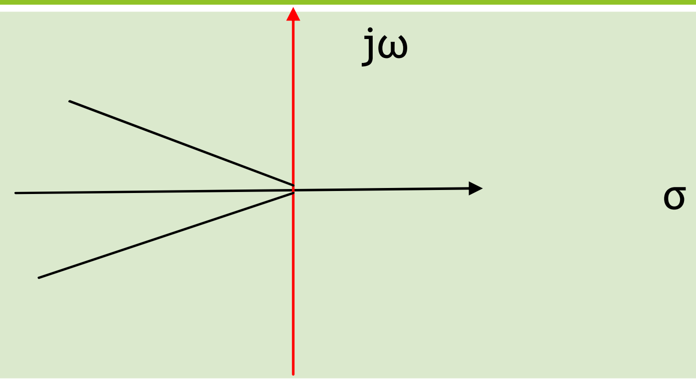
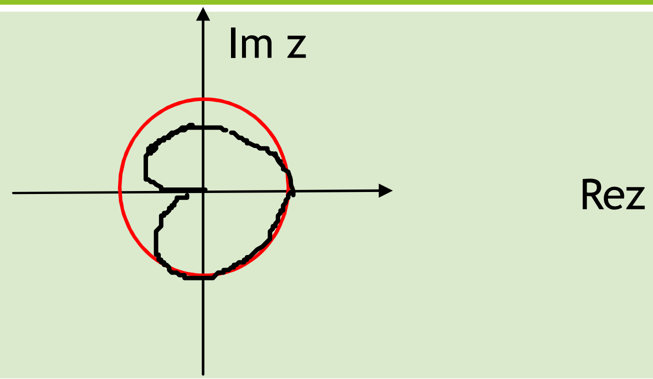
Mapping between s-plane and z-plane

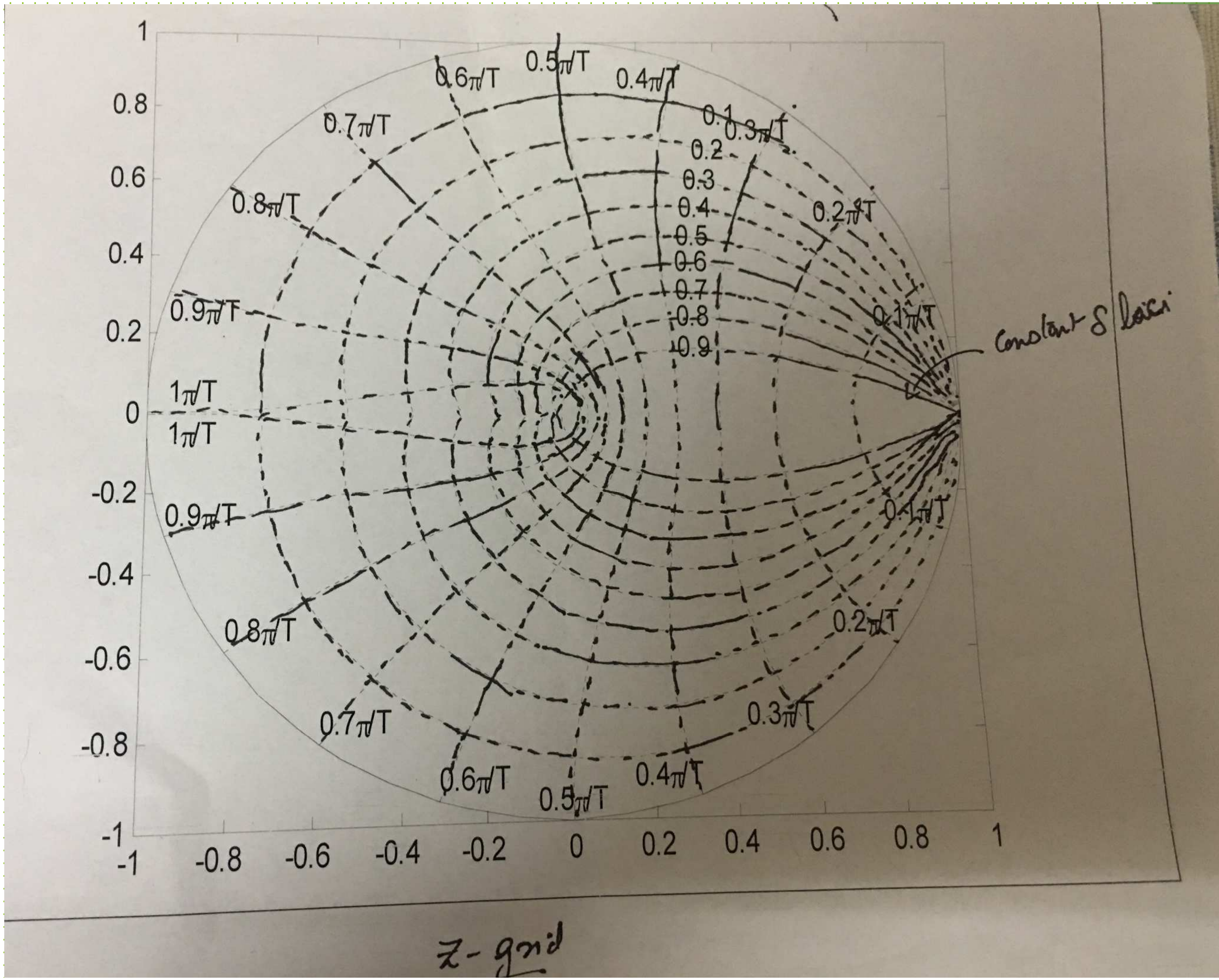
S-Plane	Z-Plane
Constant Damping Locii	
	

Mapping between s-plane and z-plane

S-Plane	Z-Plane
Constant frequency Locii	
	

Mapping between s-plane and z-plane

S-Plane Constant damping ratio Locii	Z-Plane
	



0.1π/T
0.2π/T
0.3π/T
0.4π/T
0.5π/T
0.6π/T
0.7π/T
0.8π/T
0.9π/T
1π/T
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Constant δ loci
Z-grid

Inverse z-transforms

- Partial Fraction Method
- Power series Method

Theorems of z-transform

- Addition & Subtraction
- Multiplication by a constant
- Real translation (shifting theorem)
- Complex Translation
- Partial differentiation theorem
- Initial Value theorem
- Final Value Theorem
- Real Convolution

1) Addition and Subtraction

$f_1(t) + f_2(t)$

$F_1(z) = \mathcal{Z} f_1(t) = \sum_{k=0}^{\infty} f_1(kT) z^{-k}$

$F_2(z) = \mathcal{Z} f_2(t) = \sum_{k=0}^{\infty} f_2(kT) z^{-k}$.

$f(t) = f_1(t) \pm f_2(t)$

$F(z) = F_1(z) \pm F_2(z)$

2) Multiplication by a constant

$f(t) \xrightarrow[z]{} F(z)$

$\mathcal{Z}\, a f(t) = a F(z)$

3) Real translation (shifting theorem)

$f(t) \xrightarrow[z]{} F(z)$

$\mathcal{Z} f(t - nT) = z^{-n} F(z)$

$\mathcal{Z} f(t + nT) = z^{n} [F(z) - \sum_{k=0}^{n-1} f(kT) z$

Proof

By definition

$\mathcal{Z} f(t - nT) = \sum_{k=0}^{\infty} f(kT - nT) z^{-k}$

let $k - n = m$

$k = m + n$.

$k = 0 \quad m = -n$

$k \to \infty \quad m \to \infty$

$\mathcal{Z} f(t - nT) = \sum_{m=0}^{\infty} f(mT)\, z^{-(m+n)}$

$= z^{-n} \sum_{m=0}^{\infty} f(mT) z^{-m}$

$= z^{-n} F(z)$

$\mathcal{Z} f(t$

let $k + n = m \quad k = m - n$

$k = 0 \quad m = n$

$k \to \infty \quad m \to \infty$

$\mathcal{Z} f((t + nT) = \sum_{k=0}^{\infty} f(kT + nT) z^{-k}$

$\mathcal{Z} = \underline{k}$

$$= \sum_{m=n}^{\infty} f(mT)\, z^{-(m-n)}$$

$$= z^n \left\{ \sum_{m=n}^{\infty} f(mT)\, z^{-m} \right\}$$

$$= z^n \left[\sum_{m=0}^{\infty} f(mT)\, z^{-m} - \sum_{m=0}^{n-1} f(mT) z^{-m} \right]$$

$$= z^n \left\{ F(z) - \sum_{m=0}^{n-1} f(mT)\, z^{-m} \right\}$$

) Complex translation

$$\mathcal{Z}\left[e^{\mp at} f(t) \right] = F\left[z e^{\pm aT} \right]$$

$$= \sum_{K=0}^{\infty} e^{\mp akT} f(kT)\, z^{-k}$$

$$= \sum_{K=0}^{\infty} f(kT) \left\{ e^{\pm aT} z \right\}^{-k}$$

$$= F(z_1) \qquad z_1 = e^{\pm aT} z$$

5 Partial differentiation Theorem

Let z transform of $f(t, a)$ be $F(z, a)$ — a is independent variable or Constant

$$\mathcal{Z}\left[\frac{\partial}{\partial a} f(t, a) \right] = \frac{\partial}{\partial a} F(z, a)$$

Proof

$$= \sum_{K=0}^{\infty} \frac{\partial}{\partial a} [f(kT), a)\, z^{-k}$$

$$= \frac{\partial}{\partial a} \sum_{K=0}^{\infty} f(kT, a)\, z^{-k}$$

$$= \frac{\partial}{\partial a} F(z, a)$$

6) Initial Value Theorem

If the function $f(t)$ has the z - transform $F(z)$ and if the limit $\lim_{z \to \infty} F(z)$

exists then

$$\lim_{K \to 0} f(KT) = \lim_{z \to \infty} F(z)$$

Proof

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

$$= f(0) + f(T) z^{-1} + f(2T) z^{-2} - - -$$

$$\lim_{z \to \infty} F(z) = f(0) = \lim_{K \to 0} f(kT)$$

Final value Theorem

$f(t) \to F(z)$

$(1 - z^{-1}) F(z)$ does not have pole on or outside the unit circle z-plane

$$\lim_{K \to \infty} f(kT) = \lim_{z \to 1} (1 - z^{-1}) F(z)$$

Real Convolut

$f_1(t) * f_2(t) \to F_1(z) * F_2(z)$

$f_1(t) = f_2(t) = 0 \quad t < 0$

$$F_1(z) F_2(z) = \mathcal{Z} \sum_{n=0}^{K} f_1(nT) f_2(kT - nT)$$

$$= \sum_{K=0}^{\infty} \sum_{n=0}^{K} f_1(nT) f_2(kT - nT) z^{-k}$$

$$= \sum_{n=0}^{K} f_1(nT) \sum_{K=0}^{\infty} f_2(kT - nT) z^{-K}$$

$m = k - n \qquad k = m + n$

$k = 0 \qquad m = -n$

$k \to 0 \qquad m \to \infty$

$$= \sum_{n=0}^{K} f_1(nT) z^{-n} \sum_{m=0}^{\infty} f_2(mT) z^{-m}$$

$$= F_1(z) F_2(z)$$

$$\mathcal{Z}^{-1} F_1(z) F_2(z) \neq f_1(kT) f_2(kT)$$

Limitation of z-transform

1. $Z f(t)$

$$f(t) \rightarrow f(kT) \rightarrow f(z)$$

$$Z^{-1} f(z) \neq f(t)$$

$$= f(kT)$$

2.

- The transfer function G(s) must have at least one more pole than zero or impulse response of G(s) must not have any jump discontinuity at t=0
- Otherwise response obtained by the z-transform is misleading and sometimes even incorrect